

H-TWISTED COURANT ALGEBROIDS

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Received June 2012

Revised July 1901

We generalize Hansen–Strobl’s definition of Courant algebroids twisted by a 4-form on the base manifold such that the twist H of the Jacobi identity is a four-form in the kernel of the anchor map and is closed under a naturally occurring exterior covariant derivative. We give examples and define a cohomology.

Keywords: twist of the Courant bracket; Courant algebroid; cohomology of algebroids.

1. Introduction

Courant algebroids were introduced by Liu, Weinstein, and Xu in [1] in order to describe the double of a Lie bialgebroid. They were further investigated by Roytenberg beginning during his Ph.D. studies and a formulation in terms of a Dorfman bracket was discovered [2] as well as the fitting into a two-term L_∞ -algebra [3]. In [4] Hansen and Strobl discovered four-form twisted Courant algebroids arising naturally in the Courant sigma model with a Wess–Zumino boundary term. These H -twisted Courant algebroids were further investigated by Liu and Sheng in [5] where the observation was made that exact H -twisted Courant algebroids, they fit into a short exact sequence with the tangent and cotangent bundle, always have an exact four-form H . In this paper we want to generalize the notion of H -twist and exhibit examples that do not come from an exact or even closed four-form. The idea is analogous to H -twisted Lie algebroids (introduced in [6]) that guided from an exterior covariant derivative (Proposition 6) that occurs naturally for strongly anchored almost Courant algebroids with anchor ρ on the exterior algebra of sections of $\ker \rho$, one permits the Jacobiator to be a $\ker \rho$ -four-form closed under the exterior covariant derivative. We will give examples of generalized exact four-forms, *i.e.* starting from a Courant algebroid with anchor ρ and a $\ker \rho$ -three-form with a certain integrability condition we define a Dorfman bracket together with a (non-trivial) $\ker \rho$ -four-form H that fit under the above idea.

Since already the definition of the closed generalized four-form requires sections of a possibly singular vector bundle, we also give a definition generalizing Roytenberg’s idea of Courant–Dorfman algebras in [8].

Furthermore, we carry over the idea of Stiénon and Xu [9] to define cochains as a subset of the exterior algebra of the H -twisted Courant algebroid such that the naive expression of a differential by the formula that holds for Lie algebroids actually gives a cochain again and squares to 0 in Theorem 15. We end the treatment with the obvious generalization of Dirac structures to H -twisted Courant algebroids and Strobl's as well as Sheng–Liu's idea [5] that such Dirac structures give H -twisted Lie algebroids.

In the mean-time parallel developments have shown that it is possible to simplify the definition of H -twisted Courant algebroids, see [7].

The paper is organized as follows. In Section 2 we give a short summary of the definition of Courant algebroid, two-term L_∞ -algebra introduced by Baez and Crans [10] and Roytenberg–Weinstein's observation that together with the skew-symmetric bracket the Courant algebroid gives such a two-term L_∞ -algebra. In Subsection 3.1 we begin with a definition of strongly anchored almost Courant algebroids and their natural covariant derivative on the kernel of the anchor map. We continue with the definition of H -twisted Courant algebroids and some examples. This part ends with the definition of an H -twisted Courant–Dorfman algebra. In Section 4 we define the naive cohomology of H -twisted Courant algebroids. In the last section we generalize the notion of Dirac structures and give examples of H -twisted Lie algebroids.

2. Preliminaries

Remember the definition of Courant algebroid. This goes back to Liu–Weinstein–Xu in [1]. We take the version of Roytenberg in [2, 2.6].

Definition 1. *A Courant algebroid is a vector bundle $E \rightarrow M$ together with an \mathbb{R} -bilinear (non-skewsymmetric) bracket $[\cdot, \cdot]: \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E)$, a morphism of vector bundles $\rho: E \rightarrow TM$, and a symmetric non-degenerate bilinearform $\langle \cdot, \cdot \rangle: E \otimes E \rightarrow \mathbb{R} \times M$ subject to the following axioms*

$$[\phi, [\psi_1, \psi_2]] = [[\phi, \psi_1], \psi_2] + [\psi_1, [\phi, \psi_2]], \quad (1)$$

$$[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi], \quad (2)$$

$$[\psi, \psi] = \frac{1}{2} \rho^* d\langle \psi, \psi \rangle, \quad (3)$$

$$\rho(\phi)\langle \psi, \psi \rangle = 2\langle [\phi, \psi], \psi \rangle. \quad (4)$$

where $\phi, \psi_i \in \Gamma(E)$, $f \in C^\infty(M)$, and d is the de Rham differential of the smooth manifold M .

In what follows we will identify E^* with E via the symmetric non-degenerate bilinearform $\langle \cdot, \cdot \rangle$.

From [10] we take the following definition of a two-term L_∞ -algebra.

Definition 2. *A two-term L_∞ -algebra is a two-term complex $0 \rightarrow V_1 \xrightarrow{\partial} V_0 \rightarrow 0$*

together with three more maps

$$\begin{aligned} [\cdot, \cdot]: V_0 \wedge V_0 &\rightarrow V_0, \\ \triangleright: V_0 \otimes V_1 &\rightarrow V_1, \\ l_3: V_0 \wedge V_0 \wedge V_0 &\rightarrow V_1 \end{aligned}$$

Subject to the rules

$$[\phi, \partial f] = \partial(\phi \triangleright f) \quad (5)$$

$$(\partial f) \triangleright g + (\partial g) \triangleright f = 0 \quad (6)$$

$$[\phi_1, [\phi_2, \phi_3]] + \text{cycl.} = \partial l_3(\phi_1, \phi_2, \phi_3) \quad (7)$$

$$\phi_1 \triangleright (\phi_2 \triangleright f) - \phi_2 \triangleright (\phi_1 \triangleright f) - [\phi_1, \phi_2] \triangleright f = l_3(\phi_1, \phi_2, \partial f) \quad (8)$$

$$l_3([\phi_1, \phi_2] \wedge \phi_3 \wedge \phi_4) + \phi_1 \triangleright l_3(\phi_2 \wedge \phi_3 \wedge \phi_4) + \text{unshuffles} = 0 \quad (9)$$

where $\phi_i \in V_0$ and $f \in V_1$.

As Roytenberg–Weinstein observed, the Courant algebroid gives rise to a two-term L_∞ -algebra with the identifications $V_0 = \Gamma(E)$, $V_1 = C^\infty(M)$, $\partial = l_1 = \rho^* \circ d$, $l_2(\psi_1, \psi_2) = [\psi_1, \psi_2] - \frac{1}{2}\rho^* d\langle \psi_1, \psi_2 \rangle$, $\psi \triangleright f = \frac{1}{2}\langle \psi, \partial f \rangle$, and $l_3(\psi_1, \psi_2, \psi_3) = \frac{1}{6}([\psi_1, \psi_2], \psi_3) + \text{cycl.}$

Since in the treatment of H -twisted Courant algebroids we will encounter sections of possibly singular vector bundles, we will also introduce the notion of Lie–Rinehart [11] as well as Courant–Dorfman algebras [8]. For this purpose let \mathbb{k} be a commutative ring (with unit 1) and R a commutative \mathbb{k} -algebra.

Definition 3. A Lie–Rinehart algebra $(R, \mathcal{E}, [\cdot, \cdot], \rho)$ is an R -module \mathcal{E} together with a \mathbb{k} -Lie algebra structure $[\cdot, \cdot]$ on \mathcal{E} and an R -linear representation $\rho: E \rightarrow \text{Der}(R)$ subject to the rules

$$\begin{aligned} 0 &= [\psi_1, [\psi_2, \psi_3]] + \text{cycl.}, \\ [\psi, f \cdot \phi] &= \rho(\psi)[f] \cdot \phi + f \cdot [\psi, \phi], \\ \rho[\phi, \psi] &= [\rho(\phi), \rho(\psi)]_{\text{Der}(R)}. \end{aligned}$$

Examples are \mathcal{E} the sections of a Lie algebroid $E \rightarrow M$ with $R = C^\infty(M)$.

Definition 4. Let \mathbb{k} contain $\frac{1}{2}$. A Courant–Dorfman algebra $(R, \mathcal{E}, \langle \cdot, \cdot \rangle, \rho, [\cdot, \cdot])$ consists of an R -module \mathcal{E} , a symmetric R -bilinearform $\langle \cdot, \cdot \rangle: \mathcal{E} \otimes_R \mathcal{E} \rightarrow R$, a derivation $\partial: R \rightarrow \mathcal{E}$, and a \mathbb{k} -bilinear (non-skewsymmetric) bracket $[\cdot, \cdot]: \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}$

subject to the rules

$$\begin{aligned}
[\psi, f \cdot \phi] &= \rho(\psi)[f] \cdot \phi + f \cdot [\psi, \phi], \\
\langle \psi, \partial \langle \phi, \phi \rangle \rangle &= 2 \langle [\psi, \phi], \phi \rangle, \\
[\psi, \psi] &= \frac{1}{2} \partial \langle \psi, \psi \rangle, \\
[\phi, [\psi_1, \psi_2]] &= [[\phi, \psi_1], \psi_2] + [\psi_1, [\phi, \psi_2]], \\
[\partial f, \phi] &= 0, \\
\langle \partial f, \partial g \rangle &= 0
\end{aligned}$$

for all $\phi, \psi_i \in \mathcal{E}$, $f, g \in R$. We call it almost Courant–Dorfman algebra iff only the first three rules hold.

Examples are \mathcal{E} the sections of a Courant algebroid $E \rightarrow M$, $R = C^\infty(M)$, $\partial = \rho^* \circ d$; but also Lie–Rinehart algebras with trivial pairing $\langle \cdot, \cdot \rangle \equiv 0$.

3. H -twisted Courant algebroids

3.1. Covariant derivative for strongly anchored almost Courant algebroids

Definition 5. A strongly anchored almost Courant algebroid is a vector bundle $E \rightarrow M$ together with a bilinear (non-skewsymmetric) bracket $[\cdot, \cdot]: \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E)$, a symmetric nondegenerate bilinearform $\langle \cdot, \cdot \rangle: E \otimes E \rightarrow \mathbb{R} \times M$, and a vector bundle morphism $\rho: E \rightarrow TM$, called the anchor subject to the axioms

$$\rho[\phi, \psi] = [\rho(\phi), \rho(\psi)]_{TM}, \quad (10)$$

$$[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi], \quad (11)$$

$$[\psi, \psi] = \frac{1}{2} \rho^* d \langle \psi, \psi \rangle, \quad (12)$$

$$\rho(\phi) \langle \psi, \psi \rangle = 2 \langle [\phi, \psi], \psi \rangle. \quad (13)$$

Given a smooth anchor map $\rho: E \rightarrow TM$ we define the $\Omega_M^\bullet(\ker \rho)$ to be the smooth sections $\Gamma(\wedge^\bullet E)$ that lie in the kernel of $\tilde{\rho}: \wedge^\bullet E \rightarrow TM \otimes \wedge^{\bullet-1} E: \psi_1 \wedge \psi_2 \mapsto \rho(\psi_1) \otimes \psi_2 - \rho(\psi_2) \otimes \psi_1$ and extended correspondingly for more terms.

Following an idea of Stiénon and Xu [9] we define an exterior covariant derivative on these cochains by the formula that holds for Lie algebroids.

Proposition 6. The following is an exterior covariant derivative, i.e. $C^\infty(M)$ -linear in the occurring $\psi_i \in \Gamma(M)$. For $\alpha \in \Omega_M^p(\ker \rho)$ define

$$\begin{aligned}
\langle \mathcal{D}\alpha, \psi_0 \wedge \dots \wedge \psi_p \rangle &= \sum_{i=0}^p (-1)^i \rho(\psi_i) \langle \alpha, \psi_0 \wedge \dots \wedge \hat{\psi}_i \wedge \dots \wedge \psi_p \rangle \\
&\quad + \sum_{i < j} (-1)^{i+j} \langle \alpha, [\psi_i, \psi_j] \wedge \psi_0 \wedge \dots \wedge \hat{\psi}_i \wedge \dots \wedge \hat{\psi}_j \wedge \dots \wedge \psi_p \rangle
\end{aligned} \quad (14)$$

\mathcal{D} maps $\Omega^p(\ker \rho) \rightarrow \Omega^{p+1}(\ker \rho)$ and fulfills the Leibniz rule

$$\mathcal{D}(\alpha \wedge \beta) = (\mathcal{D}\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge \mathcal{D}\beta. \quad (15)$$

Proof. The main difference to Lie algebroids is that the bracket is not skewsymmetric. However the non-skewsymmetric part of the bracket vanishes when inserted into α . The rest is now a straightforward calculation. For the last statement note that \mathcal{D} is a first order odd differential operator. \square

Note that it is also possible to split a $\ker \rho$ - $p + k$ -form α as a $\ker \rho$ - p -form with values in the k -fold exterior power of $\ker \rho$. We will denote any possible splitting as $\tilde{\alpha}$.

3.2. Definition and examples

Definition 7. An H -twisted Courant algebroid is a vector bundle $E \rightarrow M$ together with an \mathbb{R} -bilinear (non-skewsymmetric) bracket $[\cdot, \cdot]: \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E)$, a morphism of vector bundles $\rho: E \rightarrow TM$, a symmetric non-degenerate bilinear-form $\langle \cdot, \cdot \rangle: E \otimes E \rightarrow \mathbb{R} \times M$, and a $\ker \rho$ -four-form $H \in \Omega_M^4(\ker \rho)$ subject to the following axioms

$$\tilde{H}(\phi, \psi_1, \psi_2) = [\phi, [\psi_1, \psi_2]] - [[\phi, \psi_1], \psi_2] - [\psi_1, [\phi, \psi_2]], \quad (16)$$

$$\mathcal{D}H = 0, \quad (17)$$

$$[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi], \quad (18)$$

$$[\psi, \psi] = \frac{1}{2} \mathcal{D}\langle \psi, \psi \rangle, \quad (19)$$

$$\rho(\phi)\langle \psi, \psi \rangle = 2\langle [\phi, \psi], \psi \rangle. \quad (20)$$

where $\phi, \psi_i \in \Gamma(E)$, $f \in C^\infty(M)$, and \mathcal{D} is the covariant derivative defined in the previous subsection.

Lemma 8. ρ is a morphism of brackets, i.e.

$$\rho[\phi, \psi] = [\rho(\phi), \rho(\psi)]. \quad (21)$$

Proof. Start from $[\rho(\phi), \rho(\psi)][f] \cdot \chi$ for $\phi, \psi, \chi \in \Gamma(E)$, $f \in C^\infty(M)$ and expand using the Leibniz rule to iterated brackets. Then use the Jacobi identity (16), and note that the H -contributions cancel, because H is $C^\infty(M)$ -linear. \square

Example 9.

0. Courant algebroids are exactly the H -twisted Courant algebroids where $H = 0$.
1. Analogously to the H -twisted Lie algebroids we start with an untwisted Courant algebroid $(E, \langle \cdot, \cdot \rangle, \rho, [\cdot, \cdot]_0)$ and make the general ansatz

$$[\phi, \psi]_B := [\phi, \psi]_0 + \tilde{B}(\phi, \psi) \quad (22)$$

where $B \in \Omega_M^3(\ker \rho)$. The Jacobiator of this bracket is

$$\tilde{H} := \widetilde{\mathcal{D}_0 B} + \tilde{B}^2 \quad (23)$$

where $\tilde{B}^2(\psi_1, \psi_2, \psi_3) := \tilde{B}(\tilde{B}(\psi_1, \psi_2), \psi_3) + \text{cycl.}$ and the condition $\mathcal{D}H = 0$ reads as

$$0 = \mathcal{D}_B H = \mathcal{D}_0 \tilde{B}^2 + \tilde{B} \mathcal{D}_0 B + \tilde{B}^3. \quad (24)$$

In the computation we use the fact observed by Stiénon–Xu that the naive differential \mathcal{D}_0 squares to 0. If we start with a Courant algebroid with $\ker \rho$ of rank at most 4, then every $B \in \Omega_M^3(\ker \rho)$ gives a twisted Courant algebroid.

In general, if we can find nontrivial solutions of this nonlinear first order PDE, we can provide nontrivial examples of H -twisted Courant algebroids.

2. One particular case arises when we start with a Courant algebroid $(E, \rho, [\cdot, \cdot], h)$ twisted by a closed 4-form $h \in \Omega^4(M)$ in the sense of Hansen–Strobl [4]. If we pull it back to $\Omega_M^4(\ker \rho)$ via ρ^* we obtain an H -twisted Courant algebroid, because $\text{im } \rho^* \subseteq \ker \rho$ as well as

Lemma 10.

$$\mathcal{D} \circ \rho^* = \rho^* \circ d \quad (25)$$

which follows from the morphism property of the anchor map.

3. Given an H -twisted Lie algebra (an almost Lie algebra \mathfrak{g} whose Jacobi identity is twisted by a three-form with values in \mathfrak{g} and $\mathcal{D}\mathfrak{g} = 0$ for the corresponding \mathcal{D}), then this augments to an H -twisted Courant algebroid over a point iff we can find an ad-invariant symmetric bilinearform $\langle \cdot, \cdot \rangle$ for it and H is then skew-symmetric.

Proposition 11. *The H -twisted Courant algebroid $(E, \rho, [\cdot, \cdot], H)$ is a two-term L_∞ -algebra with the identifications $V_0 := \Gamma(E)$, $V_1 := \Gamma(\ker \rho)$, and the operations*

$$\partial = l_1 : V_1 \subseteq V_0, \quad (26)$$

$$l_2 : V_0 \wedge V_\bullet \rightarrow V_\bullet : (\psi_1, \psi_2) \mapsto [\psi_1, \psi_2] - \frac{1}{2} \mathcal{D} \langle \psi_1, \psi_2 \rangle, \quad (27)$$

$$l_3 : \wedge^3 V_0 \rightarrow V_1 : (\psi_1, \psi_2, \psi_3) \mapsto H(\psi_1, \psi_2, \psi_3) + \frac{1}{6} \mathcal{D} \langle [\psi_1, \psi_2], \psi_3 \rangle + \text{cycl.} \quad (28)$$

The correction in the bracket l_2 and in the Jacobiator l_3 are analogous to Roytenberg [2] and therefore fit the Courant case.

Proof. Straightforward but lengthy calculation. \square

3.3. H -twisted Courant–Dorfman algebras

Let \mathbb{k} be a commutative ring (with unit 1) that contains $\frac{1}{2}$. Analogously to Roytenberg [8] we define a strongly anchored almost Courant–Dorfman algebra as:

Definition**12. A**

strongly anchored almost Courant–Dorfman algebra $(R, \mathcal{E}, \langle \cdot, \cdot \rangle, \mathcal{D}_0, [\cdot, \cdot])$ is an R -module \mathcal{E} together with a symmetric R -bilinearform $\langle \cdot, \cdot \rangle: \mathcal{E} \otimes_R \mathcal{E} \rightarrow R$ such that $\kappa: \mathcal{E} \rightarrow \mathcal{E}^*: \psi \mapsto \langle \psi, \cdot \rangle$ is an isomorphism of R -modules, a derivation $\mathcal{D}_0: R \rightarrow \mathcal{E}$, and a \mathbb{k} -bilinear (non-skewsymmetric) bracket $[\cdot, \cdot]: \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}$ subject to the rules

$$[\psi, f \cdot \phi] = \langle \psi, \mathcal{D}_0 f \rangle \cdot \phi + f \cdot [\psi, \phi], \quad (29)$$

$$\langle \psi, \mathcal{D}_0 \langle \phi, \phi \rangle \rangle = 2\langle [\psi, \phi], \phi \rangle, \quad (30)$$

$$[\phi, \phi] = \frac{1}{2} \mathcal{D}_0 \langle \phi, \phi \rangle, \quad (31)$$

$$\langle [\psi, \phi], \mathcal{D}_0 f \rangle = \langle \phi, \mathcal{D}_0 \langle \psi, \mathcal{D}_0 f \rangle \rangle - \langle \psi, \mathcal{D}_0 \langle \phi, \mathcal{D}_0 f \rangle \rangle \quad (32)$$

Examples are \mathcal{E} the sections of a strongly anchored almost Courant algebroid $(E, \langle \cdot, \cdot \rangle, \rho, [\cdot, \cdot])$.

These strongly anchored almost Courant–Dorfman algebras inherit a derivative of degree 1 on the exterior algebra $C^p(\mathcal{E}, \mathcal{D}_0) := \mathcal{E}^{\wedge p} \cap \ker i_{\mathcal{D}_0 R}$ as before:

$$\begin{aligned} \langle \mathcal{D}\alpha, \psi_0 \wedge \dots \wedge \psi_p \rangle &:= \sum_{i=0}^p (-1)^i \langle \psi_i, \mathcal{D}_0 \langle \alpha, \psi_0 \wedge \dots \wedge \hat{\psi}_i \dots \wedge \psi_n \rangle \rangle \\ &+ \sum_{i < j} (-1)^{i+j} \langle \alpha, [\psi_i, \psi_j] \wedge \psi_0 \dots \wedge \hat{\psi}_i \dots \wedge \hat{\psi}_j \dots \wedge \psi_p \rangle \end{aligned} \quad (33)$$

Note that in particular $(\mathcal{D}|R) = \mathcal{D}_0$.

Therefore we can define H -twisted Courant–Dorfman algebras analogously to Roytenberg’s definition.

Definition 13. An H -twisted Courant–Dorfman algebra $(R, \mathcal{E}, \langle \cdot, \cdot \rangle, \mathcal{D}_0, [\cdot, \cdot], H)$ is an R -module \mathcal{E} together with a symmetric R -bilinearform $\langle \cdot, \cdot \rangle: \mathcal{E} \otimes_R \mathcal{E} \rightarrow R$ such that $\kappa: \mathcal{E} \otimes_R \mathcal{E} \rightarrow R: \psi \mapsto \langle \psi, \cdot \rangle$ is an isomorphism of R -modules, a derivative $\mathcal{D}_0: R \rightarrow \mathcal{E}$, a \mathbb{k} -bilinear (non-skewsymmetric) bracket $[\cdot, \cdot]: \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}$, and a $C^4(E, \mathcal{D}_0)$ -form H subject to the rules

$$[\psi, f \cdot \phi] = \langle \psi, \mathcal{D}_0 f \rangle \cdot \phi + f \cdot [\psi, \phi], \quad (34)$$

$$\langle \psi, \mathcal{D}_0 \langle \phi, \phi \rangle \rangle = 2\langle [\psi, \phi], \phi \rangle, \quad (35)$$

$$[\phi, \phi] = \frac{1}{2} \langle \phi, \phi \rangle, \quad (36)$$

$$\tilde{H}(\phi, \psi_1, \psi_2) = [\phi, [\psi_1, \psi_2]] - [[\phi, \psi_1], \psi_2] - [\psi_1, [\phi, \psi_2]], \quad (37)$$

$$\mathcal{D}H = 0, \quad (38)$$

$$[\mathcal{D}_0 f, \phi] = 0, \quad (39)$$

$$\langle \mathcal{D}_0 f, \mathcal{D}_0 g \rangle = 0 \quad (40)$$

where $\phi, \psi_i \in \mathcal{E}$, $f, g \in R$ and \mathcal{D} the extension of \mathcal{D}_0 as defined above.

Examples are \mathcal{E} the sections of an H -twisted Courant algebroid $(E, \langle \cdot, \cdot \rangle, \rho, [\cdot, \cdot], H)$.

4. Naive Cohomology

Proposition 14. *The covariant derivative \mathcal{D} of Subsection 3.1 does not square to 0 in general, instead it fulfills for H -twisted Courant algebroids*

$$\langle \mathcal{D}^2 f, \psi_0 \wedge \psi_1 \rangle = 0, \quad (41)$$

$$\langle \mathcal{D}^2 \phi, \psi_0 \wedge \psi_1 \rangle = H(\phi, \psi_0, \psi_1), \quad (42)$$

$$\mathcal{D}^2(\alpha \wedge \beta) = (\mathcal{D}^2 \alpha) \wedge \beta + \alpha \wedge \mathcal{D}^2 \beta \quad (43)$$

for $f \in C^\infty(M)$, $\phi \in \Gamma(\ker \rho)$, $\alpha, \beta \in \Omega_M^\bullet(\ker \rho)$, and $\psi_i \in \Gamma(E)$.

Proof. The proof is analogous to the one for H -twisted Lie algebroids, namely the first statement follows from the morphism property of ρ , the second statement is a reformulation of the Leibniz rule, and the last statement follows from the graded Leibniz rule (15). \square

Theorem 15 (Naive cohomology). *The cochains*

$$C^p(E, \rho, H) := \Omega^p(\ker \rho) \cap \ker \tilde{H} \quad (44)$$

together with the derivative

$$d: C^p(E, \rho, H) \rightarrow C^{p+1}(E, \rho, H) : \alpha \mapsto \mathcal{D}\alpha \quad (45)$$

form a cochain complex.

Proof. It remains to check that \mathcal{D} maps \tilde{H} -closed forms to \tilde{H} -closed forms. This follows from the property

$$[\mathcal{D}, \tilde{H}] = \widetilde{\mathcal{D}H} = 0 \quad (46)$$

due to the axiom (17). \square

The corresponding notion of naive cochains for Courant–Dorfman algebras is

$$C^p(\mathcal{E}, \mathcal{D}_0, H) := \ker \tilde{H}|_{\mathcal{E}^{\wedge p}} \cap \ker i_{\mathcal{D}_0 R}. \quad (47)$$

5. Dirac Structures and H -twisted Lie Algebroids

Given an H -twisted Courant algebroid (with bilinearform) of split signature, we define a Dirac structure in the usual way.

Definition 16. *Given an H -twisted Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho, H)$, we define*

1. *an isotropic subbundle $L \subseteq E$ as a vector subbundle over M such that $\langle L, L \rangle \equiv 0$. If the bilinearform is of split signature, we can consider maximal isotropic subbundles with respect to inclusion and call them Lagrangean subbundles.*

2. an integrable subbundle $L \subseteq E$ when the bracket closes on the sections of L , i.e. $[\Gamma(L), \Gamma(L)] \subseteq \Gamma(L)$.
3. a Dirac structure as a maximal isotropic integrable subbundle in an H -twisted Courant algebroid of split signature.

Compare this with the definition of H -twisted Lie algebroids (taken from [6]):

Definition 17. An H -twisted Lie algebroid is a vector bundle $E \rightarrow M$ together with a bundle map $\rho: E \rightarrow TM$ (called the anchor), a section $H \in \Omega_M^3(E, \ker \rho)$, and a skew-symmetric bracket $[\cdot, \cdot]: \Gamma(E) \wedge \Gamma(E) \rightarrow \Gamma(E)$ subject to the axioms

$$[\phi, [\psi_1, \psi_2]] = [[\phi, \psi_1], \psi_2] + [\psi_1, [\phi, \psi_2]] + H(\phi, \psi_1, \psi_2) \quad (48)$$

$$[\phi, f \cdot \psi] = \rho(\phi)[f] \cdot \psi + f \cdot [\phi, \psi] \quad (49)$$

$$DH = 0 \quad (50)$$

where $f \in C^\infty(M)$, $\phi, \psi, \psi_i \in \Gamma(E)$ and D is the one defined for anchored almost Lie algebroids analogous to (14), but ρ replaced by

$$\nabla_\psi v := [\psi, v] \quad (51)$$

for every $\psi \in \Gamma(E)$ and $v \in \Gamma(\ker \rho)$ which is an E -connection on $\ker \rho$.

We have the immediate consequence.

Proposition 18. Given an H -twisted Courant algebroid (E, H) of split signature. Then every Dirac structure $L \subseteq E$ is an H -twisted Lie algebroid. In particular the twist \tilde{H} induces a D -closed L -three-form with values in $\ker \rho|_L$.

Acknowledgments

Research on this paper was conducted during the stay at Sun Yat-sen University and partially supported by NSFC(10631050 and 10825105) and NKBRPC(2006CB805905). The paper was finished at Northwestern Polytechnical University. I am grateful to Z.-J. Liu for comments on an earlier version of this paper.

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